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The Adjoint Process in Stochastic Optimal Control

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Abstract. Using stochastic flows a minimum principle is obtained when a diffusion is controlled using stochastic open loop controls. An equation for the adjoint process is then derived using an explicit formula for the integrand in a certain stochastic integral.

1. Introduction.

There have been many proofs of minimum principles in stochastic control. For a small sample see the works of Kushner [15], Bismut [2], Haussmann [10], [11], [12], Davis and Varaiya [6], and the book by Elliott [8]. In this paper we consider a diffusion and stochastic open loop controls, that is, controls which are adapted to the filtration of the driving Brownian motion process. For such controls the dynamical equations have strong solutions, and the results on the differentiability of the solution, due originally to Blagovescenskii and Freidlin [1], can be applied. The work of Kunita [14] and Bismut [2] on stochastic flows enables the variation in the expected cost, due to a perturbation of the optimal control, to be calculated explicitly. The minimum principle follows by differentiating this quantity. another work

If the optimal control is Markov the stochastic integral representation result of [9] is applied to give an expression for a quantity associated with the adjoint process. Stochastic calculus is then used to derive the equation satisfied by the adjoint process. (KR) ←



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2. Dynamics.

Suppose the state of a system is described by a stochastic differential equation:

$$\begin{aligned} d\xi_t &= f(t, \xi_t, u)dt + g(t, \xi_t)dw_t \\ \xi_t &\in R^d, \quad \xi_0 = x_0, \quad 0 \leq t \leq T. \end{aligned} \quad (2.1)$$

The control parameter u will take values in a compact subset U of some Euclidean space R^k .

We shall make the following assumptions.

A_1 : $f : [0, T] \times R^d \times U \rightarrow R^d$ is Borel measurable, continuous in u for each (t, x) , continuously differentiable in x and for some constant K

$$(1 + |x|)^{-1} |f(t, x, u)| + |f_x(t, x, u)| \leq K_1.$$

A_2 : $g : [0, T] \times R^d \rightarrow R^d \otimes R^n$ is a matrix valued, Borel measurable function, continuously differentiable in x , and for some constant K_2

$$|g(t, x)| + |g_x(t, x)| \leq K_2.$$

The columns of g will be denoted by $g^{(k)}$ for $k = 1, \dots, n$.

A_3 : $w = (w^1, \dots, w^n)$ is an n -dimensional Brownian motion on a probability space (Ω, F, P) with a right continuous, complete filtration $\{F_t\}$, $0 \leq t \leq T$.

DEFINITION 2.1. The set of admissible controls \underline{U} will be the F_t -predictable functions on $[0, T] \times \Omega$ with values in U . These are sometimes called 'stochastic open loop' controls, [3].

REMARKS 2.2. For each $u \in \underline{U}$ there is, therefore, a strong solution of (2.1), and we shall write $\xi_{s,t}^u(x)$ for the solution trajectory given by

$$\xi_{s,t}^u(x) = x + \int_s^t f(r, \xi_{s,r}^u(x), u_r)dr + \int_s^t g(r, \xi_{s,r}^u(x))dw_r. \quad (2.2)$$

Then, because u is a (predictable) parameter, the result of Blagovenskenskii and Freidlin [1] extends to this situation, so the Jacobian $\frac{\partial \xi_{s,t}^u}{\partial x}(x) = D_{s,t}^u$ exists and is the solution of

$$D_{s,t}^u = I + \int_s^t f_x(r, \xi_{s,r}^u(x), u_r)D_{s,r}^u dr + \sum_{k=1}^n \int_s^t g_x^{(k)}(r, \xi_{s,r}^u(x))D_{s,r}^u dw_r^k. \quad (2.3)$$

Here I is the $d \times d$ identity matrix. In fact, if the coefficients f and g are C^k the map $x \rightarrow \xi_{s,t}^u(x)$ is C^{k-1} .

Consider the matrix valued process H defined by:

$$H_{s,t}^u = I - \int_s^t H_{s,r}^u (f_\ell(r, \xi_{s,r}^u(x), u_r) - \sum_{k=1}^n g_\ell^{(k)}(r, \xi_{s,r}^u(x))^2) dr - \sum_{k=1}^n \int_s^t H_{s,r}^u g_\ell^{(k)}(r, \xi_{s,r}^u(x)) dw_r^k. \quad (2.4)$$

Then using the Ito rule we see $d(H_{s,t}^u D_{s,t}^u) = 0$ and $H_{s,s}^u D_{s,s}^u = I$, so $H_{s,t}^u = (D_{s,t}^u)^{-1}$.

Write $\|\xi^u(x_0)\|_t = \sup_{0 \leq s \leq t} |\xi_{0,s}^u(x_0)|$. Then, as in Lemma 2.1 of [12], for any p , $1 \leq p < \infty$, using Gronwall's and Jensen's inequalities

$$\|\xi^u(x_0)\|_T^p \leq C \left(1 + |x_0|^p + \left| \int_0^t g(r, \xi_{0,r}^u(x_0)) dw_r \right|^p \right)$$

almost surely for some constant C . Therefore, using Burkholder's inequality and hypothesis A_2 , $\|\xi^u(x_0)\|_T$ is in L^p for all p , $1 \leq p < \infty$. Write

$$\begin{aligned} \|D^u\|_T &= \sup_{0 \leq s \leq T} |D_{0,s}^u| \\ \|H^u\|_T &= \sup_{0 \leq s \leq T} |H_{0,s}^u|. \end{aligned}$$

Then, because f_ℓ and g_ℓ are bounded, an application of Gronwall's, Jensen's and Burkholder's inequalities again implies

$$\|D^u\|_T \text{ and } \|H^u\|_T \text{ are in } L^p \text{ for all } p, \quad 1 \leq p < \infty.$$

COST 2.3. Suppose for simplicity that the cost associated with the process is purely terminal and given by a bounded C^2 function

$$c(\xi_{0,T}^u(x_0)).$$

A_4 : We suppose $|c(x)| + |c_x(x)| + |c_{xx}(x)| \leq K_3(1 + |x|^q)$ for some $q < \infty$.

The expected cost if a control $u \in \underline{U}$ is used is, therefore,

$$J(u) = E[c(\xi_{0,T}^*(x_0))].$$

We shall suppose there is an optimal control $u^* \in \underline{U}$ so

$$J(u^*) \leq J(u) \quad \text{for all } u \in \underline{U}.$$

NOTATION 2.4. If u^* is an optimal control write ξ^* for ξ^{u^*} , D^* for D^{u^*} etc.

REMARKS 2.5. Consider a d -dimensional semimartingale of the form

$$z_t = z_s + A_t$$

where A is a predictable bounded variation process. Then Kunita's formula [14] for the composition of processes can be applied, (see also Bismut [5]), and we have

$$\begin{aligned} \xi_{s,t}^*(z_t) &= z_s + \int_s^t f(r, \xi_{s,r}^*(z_r), u_r^*) dr \\ &\quad + \int_s^t \frac{\partial \xi_{s,r}^*}{\partial x}(z_r) dA_r + \sum_{k=1}^n \int_s^t g^{(k)}(r, \xi_{s,r}^*(z_r)) dw_r^k. \end{aligned} \quad (2.5)$$

DEFINITION 2.6. Consider perturbations of the optimal control u^* of the following kind: For $s \in [0, T]$, $h > 0$ such that $0 \leq s < s+h < T$, and $A \in F_s$ define, for any other admissible control $\tilde{u} \in \underline{U}$,

$$u(t, \omega) = \begin{cases} u^*(t, \omega) & \text{if } (t, \omega) \notin [s, s+h] \times A \\ \tilde{u}(t, \omega) & \text{if } (t, \omega) \in [s, s+h] \times A. \end{cases}$$

Applying (2.5) we have, similarly to Theorem 5.1 of [4], the following result.

THEOREM 2.7. For the perturbation u of u^* consider the process

$$z_t = x + \int_s^t \left(\frac{\partial \xi_{s,r}^*}{\partial x}(z_r) \right)^{-1} (f(r, \xi_{s,r}^*(z_r), u_r) - f(r, \xi_{s,r}^*(z_r), u_r^*)) dr. \quad (2.6)$$

Then the process $\xi_{s,t}^*(z_t)$ is indistinguishable from $\xi_{s,t}^{\#}(x)$.

PROOF. Substituting (2.6) in (2.5) we see

$$\begin{aligned}\xi_{s,t}^*(z_t) &= x + \int_s^t f(r, \xi_{s,r}^*(z_r), u_r^*) dr \\ &\quad + \int_s^t \left(\frac{\partial \xi_{s,r}^*(z_r)}{\partial x} \right) \left(\frac{\partial \xi_{s,r}^*(z_r)}{\partial x} \right)^{-1} \left(f(r, \xi_{s,r}^*(z_r), u_r) - f(r, \xi_{s,r}^*(z_r), u_r^*) \right) dr \\ &\quad + \int_s^t g(r, \xi_{s,r}^*(z_r)) dw_r \\ &= x + \int_s^t f(r, \xi_{s,r}^*(z_r), u_r) dr + \int_s^t g(r, \xi_{s,r}^*(z_r)) dw_r.\end{aligned}$$

However, the solution to (2.2) is unique so $\xi_{s,t}^*(z_r) = \xi_{s,t}^{\#}(x)$.

REMARKS 2.8. Note that $u(t) = u^*(t)$ if $t > s + h$ so $z_t = z_{s+h}$ if $t > s + h$.

Therefore

$$\xi_{s,t}^*(z_t) = \xi_{s,t}^*(z_{s+h}) = \xi_{s+h,t}^*(\xi_{s,s+h}^{\#}(x))$$

if $t > s + h$.

3. A Minimum Principle.

Now

$$\begin{aligned} J(u^*) &= E[c(\xi_{0,T}^*(x_0))] \\ &= E[c(\xi_{s,T}^*(x))] \quad \text{where } x = \xi_{0,s}(x_0), \end{aligned}$$

because, by uniqueness, $\xi_{0,T}^*(x_0) = \xi_{s,T}^*(x)$. Similarly,

$$\begin{aligned} J(u) &= E[c(\xi_{0,T}^u(x_0))] \\ &= E[c(\xi_{s,T}^u(x))] \\ &= E[c(\xi_{s,T}^u(x_{s+h}))]. \end{aligned}$$

Therefore,

$$J(u) - J(u^*) = E[c(\xi_{s,T}^*(x_{s+h})) - c(\xi_{s,T}^*(x))].$$

Because $\xi_{s,T}^*(\cdot)$ is differentiable this is

$$= E\left[\int_s^{s+h} c_\ell(\xi_{s,T}^*(z_r)) \frac{\partial \xi_{s,T}^*(z_r)}{\partial x} \cdot \left(\frac{\partial \xi_{s,r}^*(z_r)}{\partial x}\right)^{-1} (f(r, \xi_{s,r}^*(z_r), u_r) - f(r, \xi_{s,r}^*(z_r), u_r^*)) dr\right]. \quad (3.1)$$

This gives an explicit formula for the change in the cost resulting from a 'strong' variation in the optimal control. It involves only a time integration. The only remaining problem is to justify the differentiation of the right hand side of (3.1).

$$\text{Write } \Gamma(s, r, z_r) = c_\ell(\xi_{s,T}^*(z_r)) \frac{\partial \xi_{s,T}^*(z_r)}{\partial x} \left(\frac{\partial \xi_{s,r}^*(z_r)}{\partial x}\right)^{-1}.$$

Then

$$\begin{aligned} J(u) - J(u^*) &= \int_s^{s+h} E\left[(\Gamma(s, r, z_r) - \Gamma(s, r, x))(f(r, \xi_{s,r}^*(z_r), u_r) - f(r, \xi_{s,r}^*(z_r), u_r^*))\right] dr \\ &\quad + \int_s^{s+h} E\left[(\Gamma(s, r, x) - \Gamma(r, r, x))(f(r, \xi_{s,r}^*(z_r), u_r) - f(r, \xi_{s,r}^*(z_r), u_r^*))\right] dr \\ &\quad + \int_s^{s+h} E\left[\Gamma(r, r, x)(f(r, \xi_{s,r}^*(z_r), u_r) - f(r, \xi_{s,r}^*(z_r), u_r^*) \right. \\ &\quad \quad \left. - f(r, \xi_{s,r}^*(x), u_r) + f(r, \xi_{s,r}^*(x), u_r^*))\right] dr \\ &\quad + \int_s^{s+h} E\left[\Gamma(r, r, x)(f(r, \xi_{0,r}^*(x_0), u_r) - f(r, \xi_{0,r}^*(x_0), u_r^*))\right] dr \\ &= I_1(h) + I_2(h) + I_3(h) + I_4(h), \quad \text{say.} \end{aligned}$$

Now,

$$\begin{aligned}
 |I_1(h)| &\leq K_4 \int_s^{s+h} E \left[|\Gamma(s, r, z_r) - \Gamma(s, r, x)| (1 + \|\xi^u(x_0)\|_{s+h}) \right] dr \\
 &\leq K_4 h \sup_{s \leq r \leq s+h} E \left[|\Gamma(s, r, z_r) - \Gamma(s, r, x)| (1 + \|\xi^u(x_0)\|_{s+h}) \right] \\
 |I_2(h)| &\leq K_5 \int_s^{s+h} E \left[|\Gamma(s, r, x) - \Gamma(r, r, x)| (1 + \|\xi^u(x_0)\|_{s+h}) \right] dr \\
 &\leq K_5 h \sup_{s \leq r \leq s+h} E \left[|\Gamma(s, r, x) - \Gamma(r, r, x)| (1 + \|\xi^u(x_0)\|_{s+h}) \right] \\
 |I_3(h)| &\leq K_6 \int_s^{s+h} E \left[|\Gamma(r, r, x)| \|x - z_r\| \right] dr \\
 &\leq K_6 h \sup_{s \leq r \leq s+h} E \left[|\Gamma(r, r, x)| \|x - z_r\|_{s+h} \right].
 \end{aligned}$$

The differences $|\Gamma(s, r, z_r) - \Gamma(s, r, x)|$, $|\Gamma(s, r, x) - \Gamma(r, r, x)|$ and $\|x - z_r\|_{s+h}$ are all uniformly bounded in some L^p , $p > 1$, and

$$\lim_{r \rightarrow s} |\Gamma(s, r, z_r) - \Gamma(s, r, x)| = 0 \quad \text{a.s.}$$

$$\lim_{r \rightarrow s} |\Gamma(s, r, x) - \Gamma(r, r, x)| = 0 \quad \text{a.s.}$$

$$\lim_{h \rightarrow 0} \|x - z_r\|_{s+h} = 0.$$

Therefore,

$$\lim_{r \rightarrow s} \|\Gamma(s, r, z_r) - \Gamma(s, r, x)\|_p = 0$$

$$\lim_{r \rightarrow s} \|\Gamma(s, r, x) - \Gamma(r, r, x)\|_p = 0$$

$$\text{and } \lim_{h \rightarrow 0} \|(\|x - z_r\|_{s+h})\|_p = 0 \quad \text{for some } p.$$

Consequently, $\lim_{h \rightarrow 0} h^{-1} I_k(h) = 0$, for $k = 1, 2, 3$.

The only remaining problem concerns the differentiability of

$$I_4(h) = \int_s^{s+h} E \left[\Gamma(r, r, x) (f(r, \xi_{0,r}^*(x_0), u_r) - f(r, \xi_{0,r}^*(x_0), u_r^*)) \right] dr.$$

The integrand is almost surely in $L^1([0, T])$ so $\lim_{h \rightarrow 0} h^{-1} I_4(h)$ exists for almost every $s \in [0, T]$. However, the set of times $\{s\}$ where the limit may not exist might depend on the

control u . Consequently we must restrict the perturbations u of the optimal control u^* to perturbations from a countable dense set of controls. In fact:

- 1) Because the trajectories are, almost surely, continuous, F_ρ is countably generated by sets $\{A_{i,\rho}\}$, $i = 1, 2, \dots$ for any rational number $\rho \in [0, T]$. Consequently F_t is countably generated by the sets $\{A_{i,\rho}\}$, $\rho \leq t$.
- 2) Let G_t denote the set of measurable functions from (Ω, F_t) to $U \subset R^k$. (If $u \in \mathcal{U}$ then $u(t, \omega) \in G_t$.) Using the L^1 -norm, as in [7], there is a countable dense subset $H_\rho = \{u_{j,\rho}\}$ of G_ρ , for rational $\rho \in [0, T]$. If $H_t = \bigcup_{\rho \leq t} H_\rho$ then H_t is a countable dense subset of G_t . If $u_{j,\rho} \in H_\rho$ then, as a function constant in time, $u_{j,\rho}$ can be considered as an admissible control over any time interval $[t, T]$ for $t \geq \rho$.
- 3) The countable family of perturbations is obtained by considering sets $A_{i,\rho} \in F_t$, functions $u_{j,\rho} \in H_t$, where $\rho \leq t$, and defining as in 3.1

$$u_{j,\rho}^*(s, \omega) = \begin{cases} u^*(s, \omega) & \text{if } (s, \omega) \notin [t, T] \times A_{i,\rho} \\ u_{j,\rho}(s, \omega) & \text{if } (s, \omega) \in [t, T] \times A_{i,\rho}. \end{cases}$$

Then for each i, j, ρ

$$\lim_{h \rightarrow 0} h^{-1} \int_s^{s+h} E \left[\Gamma(r, r, x) (f(r, \xi_{0,r}^*(x_0), u_{j,\rho}^*) - f(r, \xi_{0,r}^*(x_0), u^*)) \right] dr \quad (3.2)$$

exists and equals

$$E \left[\Gamma(s, s, x) (f(s, \xi_{0,s}^*(x_0), u_{j,\rho}) - f(s, \xi_{0,s}^*(x_0), u^*)) I_{A_{i,\rho}} \right]$$

for almost all $s \in [0, T]$.

Therefore, considering this perturbation we have

$$\begin{aligned} \lim_{h \rightarrow 0} h^{-1} (J(u_{j,\rho}^*) - J(u^*)) &= E \left[\Gamma(s, s, x) (f(s, \xi_{0,s}^*(x_0), u_{j,\rho}) - f(s, \xi_{0,s}^*(x_0), u^*)) I_{A_{i,\rho}} \right] \\ &\geq 0 \quad \text{for almost all } s \in [0, T]. \end{aligned}$$

Consequently there is a set $S \subset [0, T]$ of zero Lebesgue measure such that, if $s \notin S$, the limit in (3.2) exists for all i, j, ρ , and gives

$$E \left[\Gamma(s, s, x) (f(s, \xi_{0,s}^*(x_0), u_{j,\rho}) - f(s, \xi_{0,s}^*(x_0), u^*)) I_{A_{i,\rho}} \right] \geq 0.$$

Using the monotone class theorem, and approximating an arbitrary admissible control $u \in \underline{U}$ we can deduce that if $s \notin S$

$$E \left[\Gamma(s, s, x) (f(s, \xi_{0,s}^*(x_0), u) - f(s, \xi_{0,s}^*(x_0), u^*)) I_A \right] \geq 0 \quad \text{for any } u \in U \text{ and } A \in F_s. \quad (3.3)$$

Write

$$p_s(x) = E \left[c_\ell(\xi_{0,T}^*(x_0)) \frac{\partial \xi_{s,T}^*(x)}{\partial x} \mid F_s \right] = E[\Gamma(s, s, x) \mid F_s] \quad (3.4)$$

where, as before, $x = \xi_{0,s}^*(x_0)$. Then $p_s(x)$ is the adjoint variable and we have in (3.3) proved the following minimum principle:

THEOREM 5.1. *If $u^* \in \underline{U}$ is an optimal control there is a set $S \subset [0, T]$ of zero Lebesgue measure such that if $s \notin S$*

$$p_s(x) f(s, x, u^*) \leq p_s(x) f(s, x, u) \quad \text{a.s.}$$

That is, the optimal control u^* almost surely minimizes the Hamiltonian and the adjoint variable is $p_s(x)$.

REMARKS 3.2. Under certain conditions the minimum cost attainable under the stochastic open loop controls is equal to the minimum cost attainable under the Markov, feedback controls of the form $u(s, \xi_{0,s}^*(x_0))$. See for example [2], [10]. If u_M is a Markov control, with a corresponding, possibly weak, solution trajectory ξ^{u_M} , then u_M can be considered as a stochastic open loop control $u_M(w)$ by putting

$$u_M(w) = u_M(s, \xi_{0,s}^{u_M}(x_0, w)).$$

This means the control in effect 'follows' its original trajectory ξ^{u_M} than any new trajectory. That is the control is similar to the adjoint strategies considered by Krylov [13]. The significance of this is that when we consider variations in the state trajectory ξ , and derivatives of the map $x \rightarrow \xi_{s,t}(x)$, the control does not react, and so we do not introduce derivatives in the u variable.

If the optimal control u^* is Markov the process ξ^* is Markov and

$$\begin{aligned} p_s(x) &= E[\Gamma(s, s, x) | F_s] \\ &= E[\Gamma(s, s, x) | x]. \end{aligned} \quad (3.5)$$

4. The Adjoint Process.

Suppose the optimal control u^* is Markov. As noted above, u^* can and will be considered as an open loop control. The Jacobian $\frac{\partial \xi_{s,T}^*}{\partial x}$ exists, as does $\left(\frac{\partial \xi_{s,T}^*}{\partial x}\right)^{-1}$ and higher derivatives.

THEOREM 4.1. Suppose the optimal control u^* is Markov. Then

$$\begin{aligned} p_s(x) &= E[c_\ell(\xi_{0,T}^*(x_0))D_{0,T}] - \int_0^s p_r(\xi_{0,r}^*(x_0))f_\ell(r, \xi_{0,r}^*(x_0), u_r^*)dr \\ &\quad + \int_0^s p_x(r, \xi_{0,r}^*(x_0))g(r, \xi_{0,r}^*(x_0))dw_r \\ &\quad - \int_0^s p_x(r, \xi_{0,r}^*(x_0))g(r, \xi_{0,r}^*(x_0))g_\ell(r, \xi_{0,r}^*(x_0))dr. \end{aligned}$$

PROOF. Write $f_\ell(r)$ for $f_\ell(r, \xi_{0,r}^*(x_0), u_r^*)$ and $g(r)$ for $g(r, \xi_{0,r}^*(x_0))$, etc. By uniqueness of the solutions to (2.1)

$$\xi_{0,T}^*(x_0) = \xi_{s,T}^*(\xi_{0,s}^*(x_0)) \quad (4.1)$$

so, differentiating,

$$D_{0,T} = D_{s,T} D_{0,s} \quad (4.2)$$

where $D_{0,T} = D_{0,T}^*$ etc. (without the *).

From (3.4) and (3.5)

$$p_s(x) = E[c_\ell(\xi_{0,T}^*(x_0))D_{s,T} | F_s]$$

so from (4.2)

$$p_s(x)D_{0,s} = E[c_\ell(\xi_{0,T}^*(x_0))D_{0,T} | F_s] \quad (4.3)$$

and this is a $(P, \{F_t\})$ martingale. Write $x = \xi_{0,s}^*(x_0)$, $D = D_{0,s}$. From the martingale representation result [9], the integrand in the representation of $p_s(x)D$ as a stochastic integral is obtained by the Ito rule, noting that only the stochastic integral terms will appear. These involve the derivatives in x and D . Therefore

$$\begin{aligned} p_s(x)D &= E[c_\xi(\xi_{0,T}^*(x_0))D_{0,T}] + \int_0^s p_x(r, \xi_{0,r}^*(x_0))g(r)dw_r D_{0,r} \\ &\quad + \sum_{k=1}^n \int_0^s p_r(\xi_{0,r}^*(x_0))g_\xi^{(k)}(r)D_{0,r}dw_r^k. \end{aligned} \quad (4.4)$$

Recall from (2.4) that $H_{0,s} = D^{-1}$ so forming the product of (2.4) and (4.4), using the Ito rule:

$$\begin{aligned} p_s(x) &= (p_s(x)D)H_{0,s} \\ &= E[c_\xi(\xi_{0,T}^*(x_0))D_{0,T}] - \int_0^s p_r(\xi_{0,r}^*(x_0))f_\xi(r)dr \\ &\quad - \sum_{k=1}^n \int_0^s p_r(\xi_{0,r}^*(x_0))g_\xi^{(k)}(r)dw_r^k + \sum_{k=1}^n \int_0^s p_r(\xi_{0,r}^*(x_0))(g_\xi^{(k)}(r))^2 dr \\ &\quad + \int_0^s p_x(r, \xi_{0,r}^*(x_0))g(r)dw_r + \sum_{k=1}^n \int_0^s p_r(\xi_{0,r}^*(x_0))g_\xi^{(k)}(r)dw_r^k \\ &\quad - \sum_{k=1}^n \int_0^s p_x(r, \xi_{0,r}^*(x_0))g(r)g_\xi^{(k)}(r)dr - \sum_{k=1}^n \int_0^s p_r(\xi_{0,r}^*(x_0))(g_\xi^{(k)}(r))^2 dr \\ &= E[c_\xi(\xi_{0,T}^*(x_0))D_{0,T}] - \int_0^s p_r(\xi_{0,r}^*(x_0))f_\xi(r)dr \\ &\quad + \int_0^s p_x(r, \xi_{0,r}^*(x_0))g(r)dw_r - \sum_{k=1}^n \int_0^s p_x(r, \xi_{0,r}^*(x_0))g(r)g_\xi^{(k)}(r)dr \end{aligned}$$

so establishing the result.

This verifies by a simple, direct method the formula of Haussman [10].

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